A Formula for Multi-Loop 4-Particle Amplitude in Superstring Theory

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Abstract

Based on the recent developments of explicit computations at 2 loops in superstring theory in the covariant RNS formalism, we propose an explicit formula for the arbitrary loop 4-particle amplitude in superstring theory. We prove that this formula passes two very difficult tests: modular invariance and factorization. If proved, this shows that superstring theory is not only finite order by order in perturbation theory but is also exceptionally simple.

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1 Introduction

String perturbation theory (for a general review see [1]) was a very active research area after the first superstring revolution in 1984/85 [2, 3, 4]. The general belief is that superstring theory gives a finite theory of quantum gravity order by order in perturbation theory [5]. Although we now understood this in general terms [6], the details have never been clearly spelled out in full generality [7, 8, 9]. The main difficulty is due to the problem of covariant quantization of superstring theory. The Green-Schwarz superstring [10] is a highly nonlinear theory. Only by choosing the light-cone gauge we have a free theory on the world-sheet. It then becomes quite easy to compute tree- and one-loop amplitudes [11, 12]. Due to the choice of the non-covariant light-cone gauge (and also the complicated dependence on interaction points) it is quite difficult if not impossible to compute multi-particle and higher-loop amplitudes.

The other approach is the covariant Ramond-Neveu-Schwarz (RNS) formalism [13, 14]. Here we can use the full power of super-conformal field theory. The problem with this formalism is that we must sum over all spin structures to get a consistent and/or supersymmetric theory [15, 16]. Also the naive integration over super-moduli space gives a result which depends on the insertion points of the super-Beltrami differentials [17]. So the results depend on these spurious unphysical poles. Although these poles are harmless to the physical amplitudes (they are total derivatives in moduli space), they give complications to the analysis of the finiteness property of the physical amplitudes.

At two loops the problem with modular invariance was solved by Gava, Iengo and Sotkov in [18]. They showed that modular invariance at two loops fixed a unique way of summation over spin structures and proved the vanishing of the cosmological constant. Iengo and the author then used this result to show the nonrenormalization theorem and to compute explicitly the non-vanishing 4-particle in superstring theory [19]. As expected the result (before integration over moduli space) depends on the choice of the insertion points. It is proved that changing the insertion points indeed give a total derivative in moduli space. The apparent contradiction with the R^4 conjecture [20, 21] was solved by recasting the amplitude into an explicitly modular invariant form [22] and the 2-loop correction to R^4 term was indeed computed to be vanishing [23].

Recently the problem of the arbitrariness at 2 loops was finally solved by D' Hoker and Phong in a series of papers [24]. Basically D' Hoker and Phong started from first principles and gauge slice independence are kept at every stage of the evaluation by using the method of projection onto super period matrices. A gauge slice independent new measure was obtained. Explicit computations were then carried out by using this new measure and indeed a gauge slice independent result was obtained [25]. This result was also derived more rigorously in [26]. Recently D' Hoker and Phong [27] also gave a measure for three loop superstring theory. It remains to see if this can be used to do explicit three loop computations in superstring theory.

Another promising approach of covariant quantization of superstring is Berkovits' pure spinor approach [28, 29, 30] or closely related ones

[31, 32, 33, 34]. Recently this has been used for proving the general non-renormalization theorem [35]. Also the leading contribution of the 4-particle amplitude is shown to be 0, in agreement with the R^4 conjecture. It remains to see how this method can be used to compute the full non-vanishing 4-particle amplitude.

Explicit result for higher loop amplitudes in superstring theory is quite rare and so the known result should be exploited to obtain as much information as possible. To our knowledge the only explicitly known higher loop (≥ 2) non-vanishing amplitude is the 2-loop four-particle amplitude in superstring theory, firstly obtained in [19] and later re-obtained in [25, 26] in an explicitly gauge independent way. Although this result is somehow quite unique in itself, it nevertheless shows some surprising simplicity and pattern which may be generalized straightforwardly to higher-loops. The first pattern is a complete cancellation of determinant factors after summation over spin structures. The second pattern is the appearance of holomorphic abelian differentials in the integrand. This appearance of holomorphic abelian differentials is also found in the explicit multi-loop computations of Antoniadis, Gava, Narain and Taylor in [36]. Although they computed a much simpler amplitude which reduces to the topological string amplitude, we found the appearance of the determinant of the abelian differentials tantalizing. We will generalize these two patterns to multi-loops and propose an explicit formula for the arbitrary loop 4-particle amplitude in superstring theory. To our surprise, this formula can pass two very difficult tests: modular invariance and factorization. We hope that this formula will be proved someday. If finally proved, this shows that superstring theory is not only finite order by order in perturbation theory but is also exceptionally simple. Generalization of the formula to heterotic string theory may be easily found and it may have far reaching consequence for the computations of higher loop amplitudes in the maximal supersymmetric gauge theory [37, 38].

2 The 4-particle amplitude in superstring theory

Before we gave our formula for multi-loops, let us first recall the 2-loop 4-particle amplitude obtained in [19, 25, 26] is:

$$A_4^{2-\text{loop}} = K \int_{\mathcal{M}_g} \frac{|\prod_{I\leq J}^2 \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma^4} |\mathcal{Y}_s|^2 \exp\Bigg(- \sum_{i<j} k_i \cdot k_j \, G(z_i, z_j)\Bigg), \tag{1}$$

where

$$\begin{aligned} \mathcal{Y}_S &= +(k_1 - k_2)\cdot(k_3 - k_4)\,\Delta(z_1, z_2)\Delta(z_3, z_4) \\ &\quad +(k_2 \leftrightarrow k_3, z_2 \leftrightarrow z_3) + (k_2 \leftrightarrow k_4, z_2 \leftrightarrow z_4) \\ &= (t-u)\Delta(z_1, z_2)\Delta(z_3, z_4) + (t-s)\Delta(z_1, z_3) \\ &\quad \times\Delta(z_2, z_4) + (u-s)\Delta(z_1, z_4)\Delta(z_3, z_3), \end{aligned} \tag{2}$$

$$\Delta(z, w) = \omega_1(z)\omega_2(w) - \omega_2(z)\omega_1(w), \tag{3}$$

$$G(z,w) \quad = \quad -\ln|E(z,w)|^2 + 2\pi \mathrm{Im}\int_z^w \omega_I (\mathrm{Im}\Omega)^{-1}_{IJ}\mathrm{Im}\int_z^w \omega_J. \quad (4)$$

As we said in the introduction, there is no determinant factors in (1). The integrand $\mathcal{Y}_s$ is defined only in terms of the holomorphic abelian differentials $\omega_i(z)$. In fact this two patterns can be straightforwardly generalized to arbitrary loops.

Our proposed formula for the g-loop 4-particle amplitude in superstring theory is:

$$A_4^{g-\mathrm{loop}} = K\int_{\mathcal{M}_g} \frac{|\wedge^a W^a \wedge_i W_i|^2}{(\det \mathrm{Im}\,\Omega)^5}\int_{\Sigma^4}|\mathcal{Y}_s|^2 \exp\Bigg(-\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\Bigg). \tag{5}$$

By comparing with the 2-loop expression given in eq. (1) we need to explain what is the measure $\wedge^a W^a \wedge_i W_i$. $\mathcal{Y}_s$ is the same expression given in (2) but with an appropriately generalized $\Delta(z,w)$ (denoted as $\Delta_g(z,w)$ to indicate it's for g-loop).

For a given Riemann surface, we fix an arbitrary generic 1-differential $\omega(z)$. This has $2g-2$ zeroes which we denoted as $P_1,\cdots,P_{2g-3}$ and $P_{2g-2}=Q$. To construct a basis of holomorphic 2-differentials, we also need a normalized Abelian differential of the 3rd kind which is defined to have vanishing A-period and to have simple poles at two different points P and Q with residues ± 1. Explicitly we have:

$$\omega_{PQ}(z) = d\,\ln(E(z,P)/E(z,Q)), \tag{6}$$

where $E(z,P)$ is the prime form.

By using the given $\omega(z)$, we can construct a basis for the holomorphic 2-differentials as follows [40]:

$$\phi^a \quad = \quad \omega(z)\omega_{P_aQ}(z), \quad a = 1,\cdots,2g-3, \tag{7}$$

$$\phi_i \quad = \quad \omega(z)\omega_i(z), \quad i=1,\cdots,g. \tag{8}$$

By the well-known correspondence between holomorphic 2-differentials and the holomorphic cotangent space of the moduli space of Riemann surface, we have following elements in the holomorphic cotangent space [39]:

$$W^a = k(\phi^a), \qquad W_i = k(\phi_i). \tag{9}$$

To refresh our memory we quote the well-known result:

$$k(\omega_i\omega_j) = \frac{1}{2\pi i}\mathrm{d}\Omega_{ij}, \tag{10}$$

where (Ω_{ij}) is the period matrix of the Riemann surface. By taking the wedge products of all W^a and W_i, we obtain a volume form in moduli space which can be used to integrate over the moduli space. This give the measure factor $\wedge^a W^a \wedge^i W_i$ in eq. (5).

The generalized $\Delta_g(z_i,z_j)$ is constructed also by making use the zeroes of $\omega(z)$. Setting

$$\det\omega(z_1,z_2,\cdots,z_g) \equiv \det(\omega_i(z_j)), \tag{11}$$

we defined

$$\begin{aligned}\Delta_g(z_i, z_j) &= \sum_{\sigma'(1)=1} (-1)^{\mathrm{sgn}(\sigma)} \det \omega(z_i, P_{\sigma(1)}, \cdots, P_{\sigma(g-1)}) \\ &\quad \times \det \omega(z_j, P_{\sigma(g)}, \cdots, P_{\sigma(2g-2)}) - (z_i \leftrightarrow z_j), \qquad (12)\end{aligned}$$

where the summation over all permutation σ is restricted to $\sigma(1) = 1$ and there is no summation over σ which only changes the ordering of P's within the 2 determinants. In total there are $2\times(2g-3)!/(g-1)!/(g-2)!$ different terms.

For later use we can also use eq. (5) to obtain the g-loop 3-particle amplitude with one massive tensor or 2-particle self-energy correction to massive tensor, by factorizing 2 massless particles. The result is:

$$A_n^{g-\mathrm{loop}} = K_n \int_{\mathcal{M}_g} \frac{|\wedge^a W^a \wedge_i W_i|^2}{(\det \mathrm{Im}\,\Omega)^5} \int_{\Sigma^n} |\mathcal{Y}_s^{(n)}|^2 \exp\left(-\sum_{i<j}^n k_i \cdot k_j\, G(z_i, z_j)\right), \qquad (13)$$

where

$$\mathcal{Y}_s^{(3)} = \Delta_g(z_1, z_3)\Delta_g(z_2, z_3), \qquad (14)$$

$$\mathcal{Y}_s^{(2)} = (\Delta_g(z_1, z_2))^2. \qquad (15)$$

To finish this section we note that a direct application of the proposed formula of (5) at two loops doesn't give the known 2-loop result of (1). In fact $\Delta_g(z, w) = \Delta(z, w)\Delta(P_1, P_2)$ which vanishes identically at two loops because $\Delta(P_1, P_2) = 0$ when $P_{1,2}$ are the zeroes of an abelian differential. This doesn't happen for higher loops. The 2-loop result can be obtained correctly if we perturb the zero points $P_{1,2}$. This is in fact necessary because of the special Z_2 symmetry at two loops which is manifest by using the hyperelliptic representation.

3 The modular invariance of the 4-particle amplitude

First we note that the measure factor doesn't depend on the specific choice of one of the zeroes of $Q = P_{2g-2}$. This can be easily seen by noting the following relations between ω_{PQ}:

$$\omega_{P_a Q}(z) = \omega_{P_a \tilde{Q}}(z) - \omega_{Q\tilde{Q}}(z). \qquad (16)$$

This induces a linear changes of basis of ϕ^a if we change to another zero point of $\tilde{Q} \in \{P_1, \cdots, P_{2g-3}\}$ of $\omega(z)$. So the measure factor $\wedge^a W^a$ just changes a sign.

To prove the the modular invariance of the 4-particle amplitude given in eq. (5), let us consider a modular transformation Γ_g:

$$\Gamma_g = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \quad \in Sp(2g, Z). \qquad (17)$$

The holomorphic 1-forms ω_i and the period matrix Ω transform as follows:

$$\begin{aligned} \omega_i &\to \tilde{\omega}_i = \omega_j (C\Omega + D)^{-1}_{ji}, &(18)\\ \Omega &\to \tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}. &(19) \end{aligned}$$

By using these results we have:

$$\begin{aligned} \det \tilde{\omega}_i(z_j) &= \det(C\Omega + D)^{-1} \det \omega_i(z_j), &(20)\\ \det \mathrm{Im}\tilde{\Omega} &= |\det(C\Omega + D)|^{-2} \det \mathrm{Im}\tilde{\Omega}. &(21) \end{aligned}$$

This proved that the following combination:

$$\frac{|\mathcal{Y}_s|^2}{(\det \mathrm{Im}\Omega)^4}, \tag{22}$$

is modular invariant ($g \geq 3$).

As ω doesn't depend on the choice of homology cycles used to defined the basis $\omega_i(z)$, one sees that $W^1 \wedge W^2 \wedge \cdots W^{2g-3}$ is modular invariant. On the other hand W_i transforms identically as ω_i:

$$W_i \to \tilde{W}_i = W_j (C\Omega + D)^{-1}_{ji}. \tag{23}$$

This gives

$$\wedge_i W_i \to \wedge_i \tilde{W}_i = \det(C\Omega + D)^{-1} \wedge_i W_i, \tag{24}$$

and so the following combination:

$$\frac{| \wedge^a W^a \wedge_i W_i|^2}{\det \mathrm{Im}\, \Omega}, \tag{25}$$

is also modular invariant. This proves that the 4-particle amplitude given in eq. (5) is modular invariant.

4 The factorization of the 4-particle amplitude

Now we prove that the 4-particle amplitude also satisfies the factorization condition.

To study the factorization, let us consider the dividing limit of the Riemann surface[1]. One way of taking this limit is to construct a family of degenerating surfaces near D_{g_1} over the unit disk $D = \{t \in C||t| < 1\}$ as follows [41, 42]. Take two surfaces Σ_1 and Σ_2 of genus g_1 and $g_2 = g - g_1$. Choose on each surface a point p_i, $i = 1, 2$, and a coordinate neighbourhood $U_i = \{z_i||z_i| < 1\}$ near each p_i, such that $p_i = \{z_i = 0\}$. Remove a small disk $|z_i| < |t|^{1/2}$ from both surfaces, and glue the remaining surfaces together by attaching the annulus $A_t = \{w||t|^{1/2} < |w| < |t|^{-1/2}\}$ according to

$$w = \begin{cases} \frac{t^{1/2}}{z_1} & \text{if } |t|^{1/2} < |w| < 1, \\ t^{-1/2} z_2 & \text{if } 1 < |w| < |t|^{-1/2}. \end{cases} \tag{26}$$

[1] We will not consider the other limit of pinching a non-zero homology cycle which would give a 6-particle amplitude .

We denote the resulting surface by Σ_t. The two components of $\Sigma_t - A_t$ we call X_t and Y_t respectively. The parameter t (which should not be confused with the standard Mandelstam variable $t = -(k_2+k_3)^2$), the points p_1 and p_2, together with the moduli of Σ_1 and Σ_2 provide a parametrization of M_g near D_{g_1}. Furthermore t is the correct analytical coordinate on M_g transversal to D_{g_1} for $g_1 > 1$. (Near D_1 the analytic transversal coordinate is t^2, as one can see explicitly in [43] by explicit computation. This is because the punctured surfaces at D_1 have an automorphism of order two, which means that t and $-t$ correspond to the same points on M_g.) The measure near the boundary ($t \to 0$) behaves as:

$$\wedge^a W_a \wedge_i W_i \to \wedge^a W_a^{(1)} \wedge_i W_i^{(1)} \wedge \mathrm{d}p_1 \wedge \mathrm{d}p_2 \wedge \mathrm{d}t \wedge^a W_a^{(2)} \wedge_i W_i^{(2)}. \quad (27)$$

Because all determinants cancel, there is no singular factor of t^{-2} which appears in bosonic string theory [44, 45, 46].

Choosing a generic abelian holomorphic differential $\omega(z,t)$ on Σ_t. At leading order in t, this will have $2g_1 - 2$ zeroes on X_t and $2g_2 - 2$ zeroes on Y_t. There are two more zeros on A_t which we denoted as Q_1 and Q_2. In the limit $t \to 0$, $Q_i \to p_i$. The basic relation is:

$$\Delta_g(z_1, z_2) = O(t), \quad (28)$$

if both $z_{1,2} \in X_t$, or both $z_{1,2} \in Y_t$, and

$$\begin{aligned} \Delta_g(z_1, z_2) &= \pm\Delta_{g_1}(z_1, Q_1)\Delta_{g_2}(Q_2, z_2) + O(t) \\ &= \pm\Delta_{g_1}(z_1, p_1)\Delta_{g_2}(p_2, z_2) + O(t), \end{aligned} \quad (29)$$

if $z_1 \in X_t$ and $z_2 \in Y_t$. We will not present the proof of the above results here.

By using these results we have

$$\begin{aligned} \mathcal{Y}_s &\to \pm(t+u-s)\Delta_{g_1}(z_1, p_1)\Delta_{g_2}(p_2, z_3)\Delta_{g_1}(z_1, p_1)\Delta_{g_2}(p_2, z_4) \\ &= \mp 3s\, \mathcal{Y}_s^{(3)}(z_1, z_2, p_1)\mathcal{Y}_s^{(3)}(z_3, z_4, p_2), \end{aligned} \quad (30)$$

which shows that $\mathcal{Y}_s$ factorizes holomorphically into a product of two 3-particle integrands.

Under the dividing degeneration limit, the exponential factor approaches the following limit ($\alpha' = 2$):

$$\begin{aligned} &|t|^{(k_1+k_2)^2} \langle \mathrm{e}^{ik_1\cdot X(z_1)}\mathrm{e}^{ik_2\cdot X(z_2)}\mathrm{e}^{-i(k_1+k_2)\cdot X(p_1)}\rangle \\ &\quad \times\langle \mathrm{e}^{ik_3\cdot X(z_3)}\mathrm{e}^{ik_4\cdot X(z_4)}\mathrm{e}^{i(k_1+k_2)\cdot X(p_2)}\rangle. \end{aligned} \quad (31)$$

Integrating over t gives exactly the (massive) propagator:

$$\int \mathrm{d}^2 t |t|^{(k_1+k_2)^2} \sim \frac{-2\pi}{s-2}. \quad (32)$$

This finishes the proof that the 4-particle amplitude of eq. (5) also satisfies factorization condition. Of course the above analysis should be modified for $g_1 = 1$.

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